

Economic Foundations and Applications of Risk

Part A. Foundations

Chapter 3: Measures of Risk

Till Stowasser

LMU, 2024

Stochastic Dominance (SD)

- SD is a concept that allows a preference ranking of distributions.
- While satisfying the property of **transitivity**, this concept is **not complete**, i.e. it will not be possible to rank all distributions.
- **First-order stochastic dominance (FOSD)**...
 - ...is a **very general** measure that allows a preference ordering for all utility functions u with $u' > 0$.
 - Its downside is that the **ranking is very incomplete**.
- **Second-order stochastic dominance (SOSD)**...
 - ...is **less general**, as it only holds for risk-averse individuals with $u'' < 0 < u'$.
 - It allows for a **less incomplete ranking** than FOSD, even though there will still be lotteries that cannot be generally ranked by SOSD either.
- There are concepts of **higher-order stochastic dominance**, which allow for the ranking of a vaster class of distributions, but which, in turn, require starker restrictions on utility function u .

Definition of first-order stochastic dominance

Definition 3.1: First-order stochastic dominance

- Let $F_A(x)$ and $F_B(x)$ be two continuously differentiable cumulative distribution functions.
- Then F_B is said to **first-order stochastically dominate** F_A iff

$$\forall x \in \mathbb{R} : F_B(x) \leq F_A(x)$$

and

$$\exists x \in \mathbb{R} : F_B(x) < F_A(x).$$

Comments:

- Recall that $F_B(x) \leq F_A(x) \equiv \text{Prob}_B(X \leq x) \leq \text{Prob}_A(X \leq x)$
- **Intuition:** Distribution B always has a lower probability to return a lower x than distribution A .

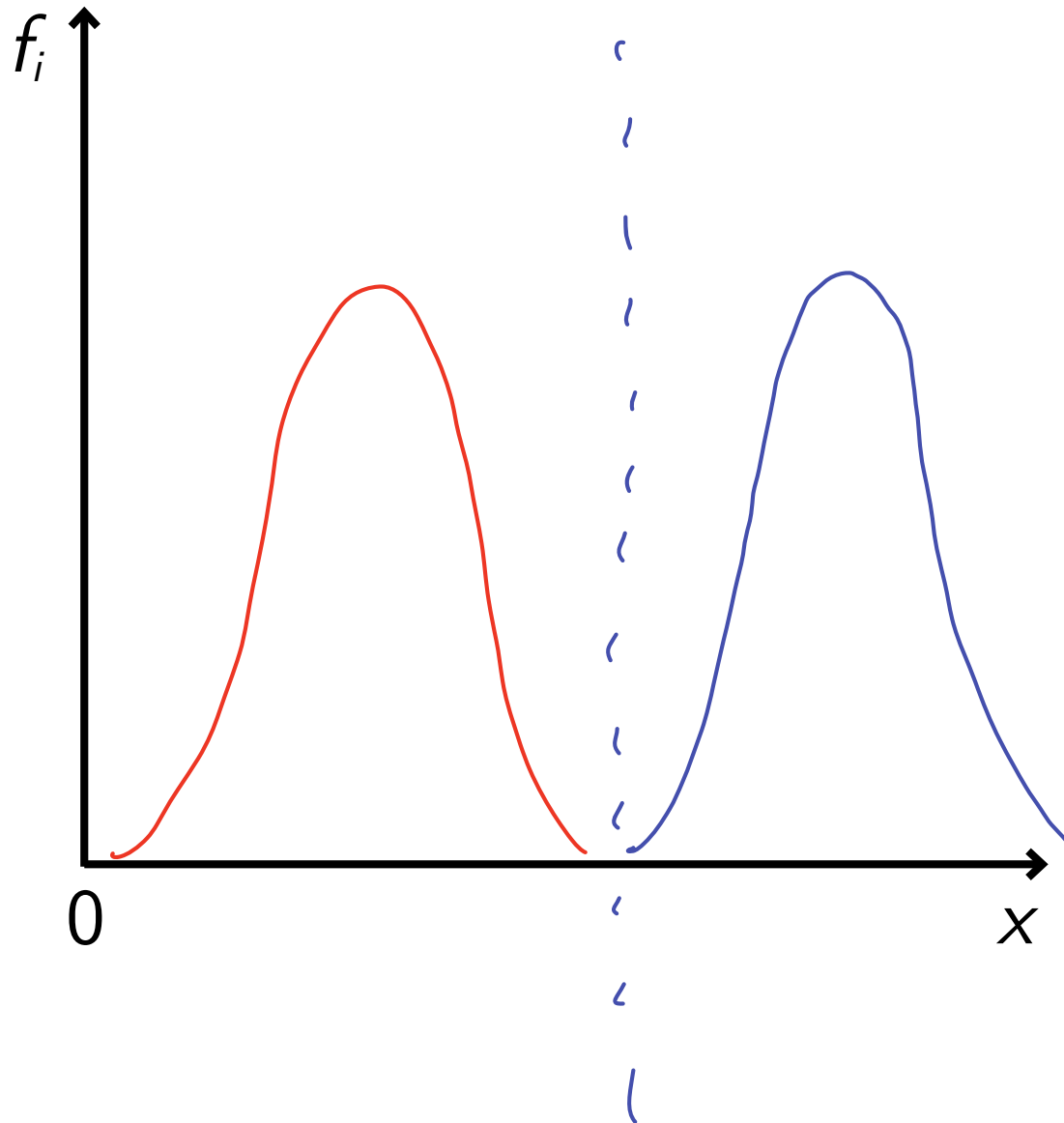
FOSD theorem

Theorem 3.1: FOSD theorem

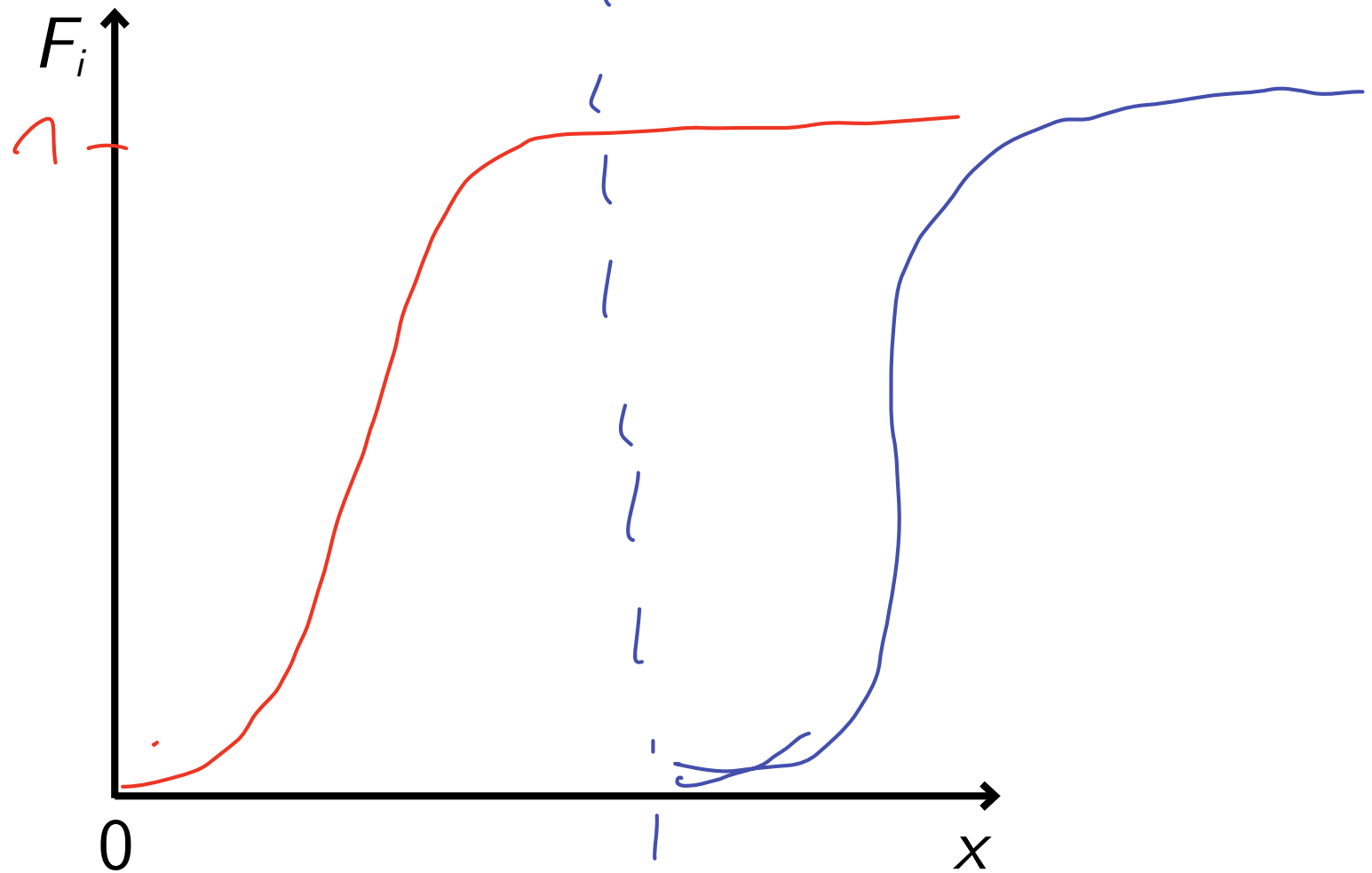
- 1 Risk-loving, risk-neutral, and risk-averse individuals with a **positive marginal utility** in income **prefer the first-order stochastically dominating distribution** of income.
- 2 This implies that FOS dominated distributions have a lower **expected value** than FOS dominating distributions (Necessary, but not sufficient condition for FOSD).

$$F_B(x) \succ^{FOSD} F_A(x) \implies E_B[x] > E_A[x]$$

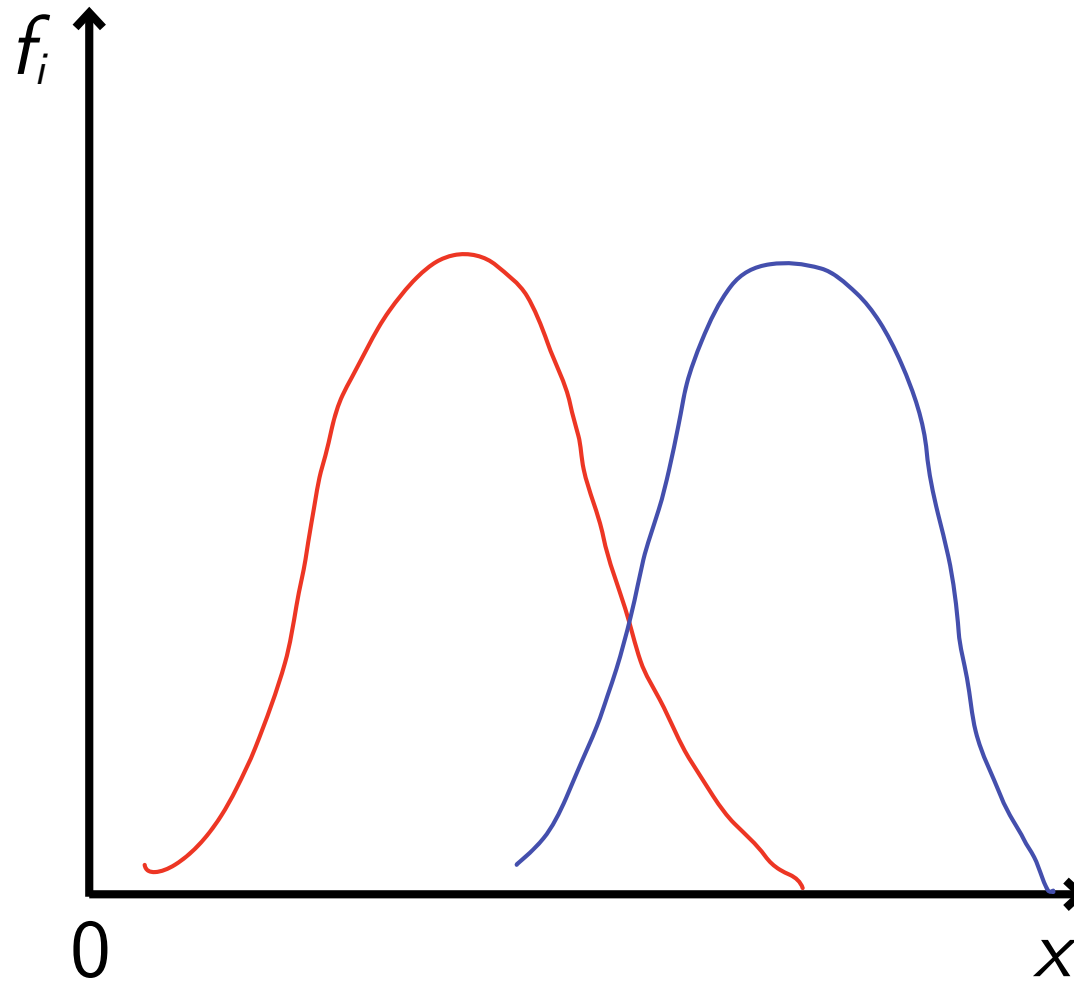
Example for strict domination: Densities



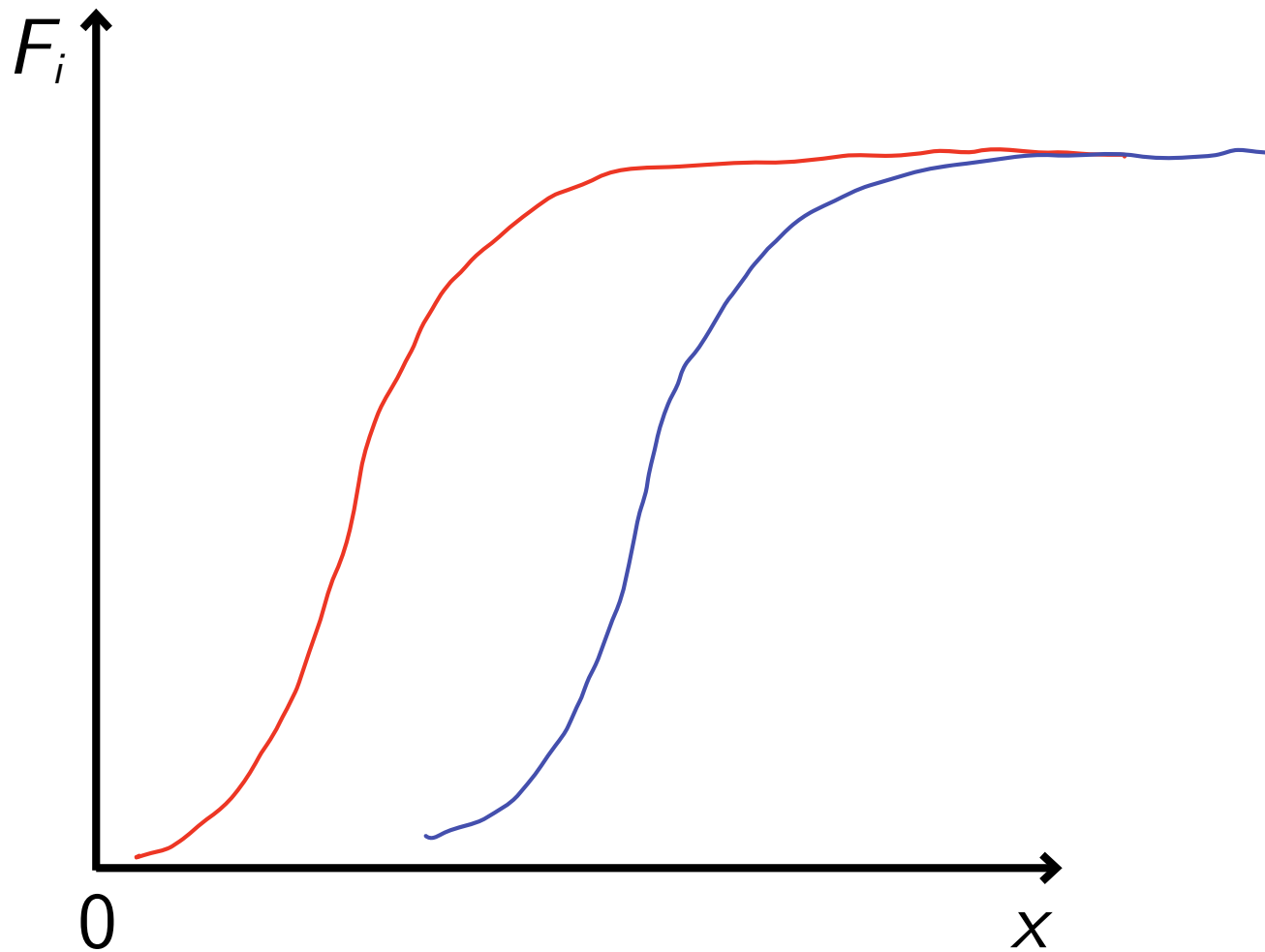
Example for strict domination: CDFs



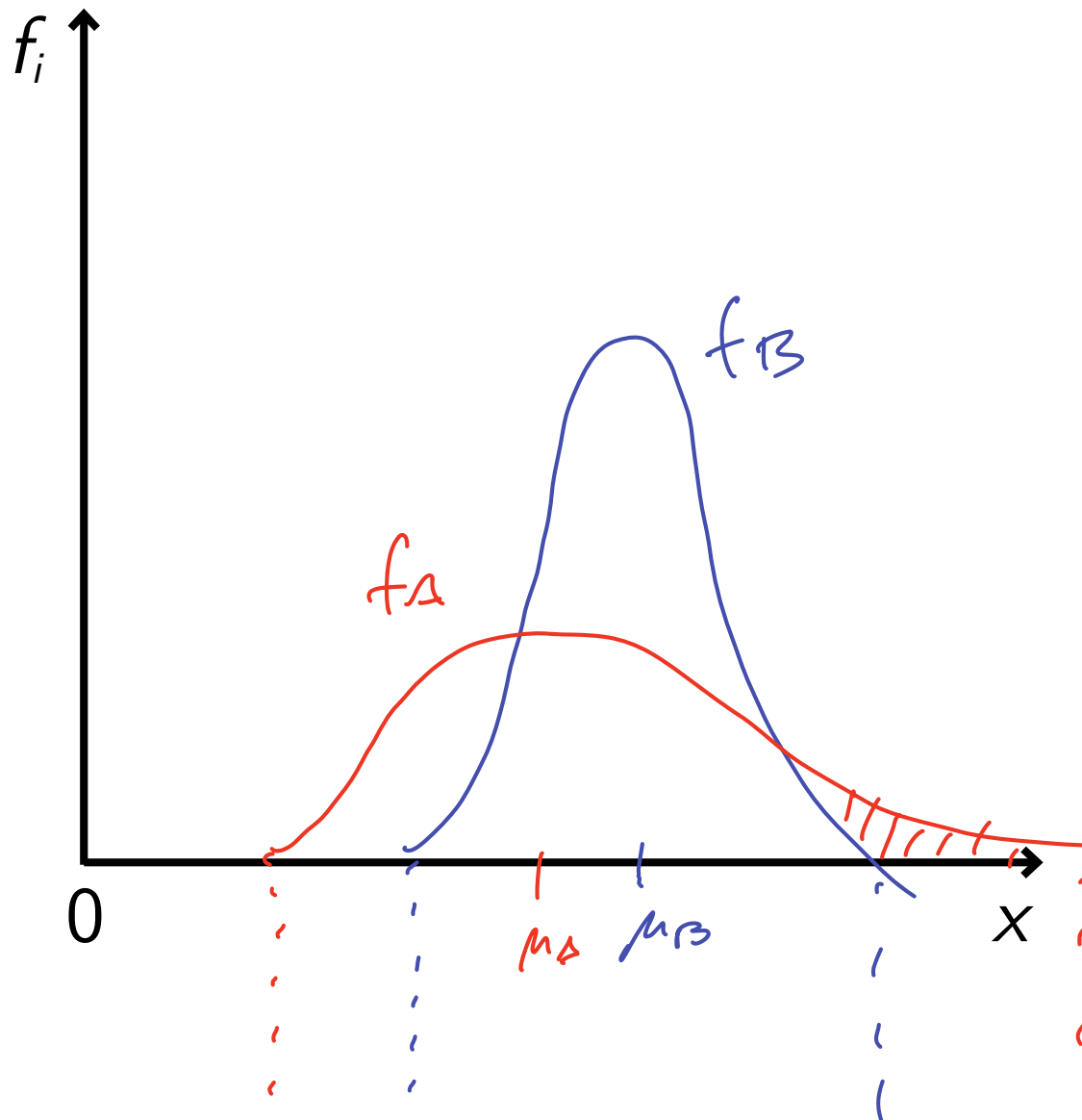
Example for FOS domination: Densities



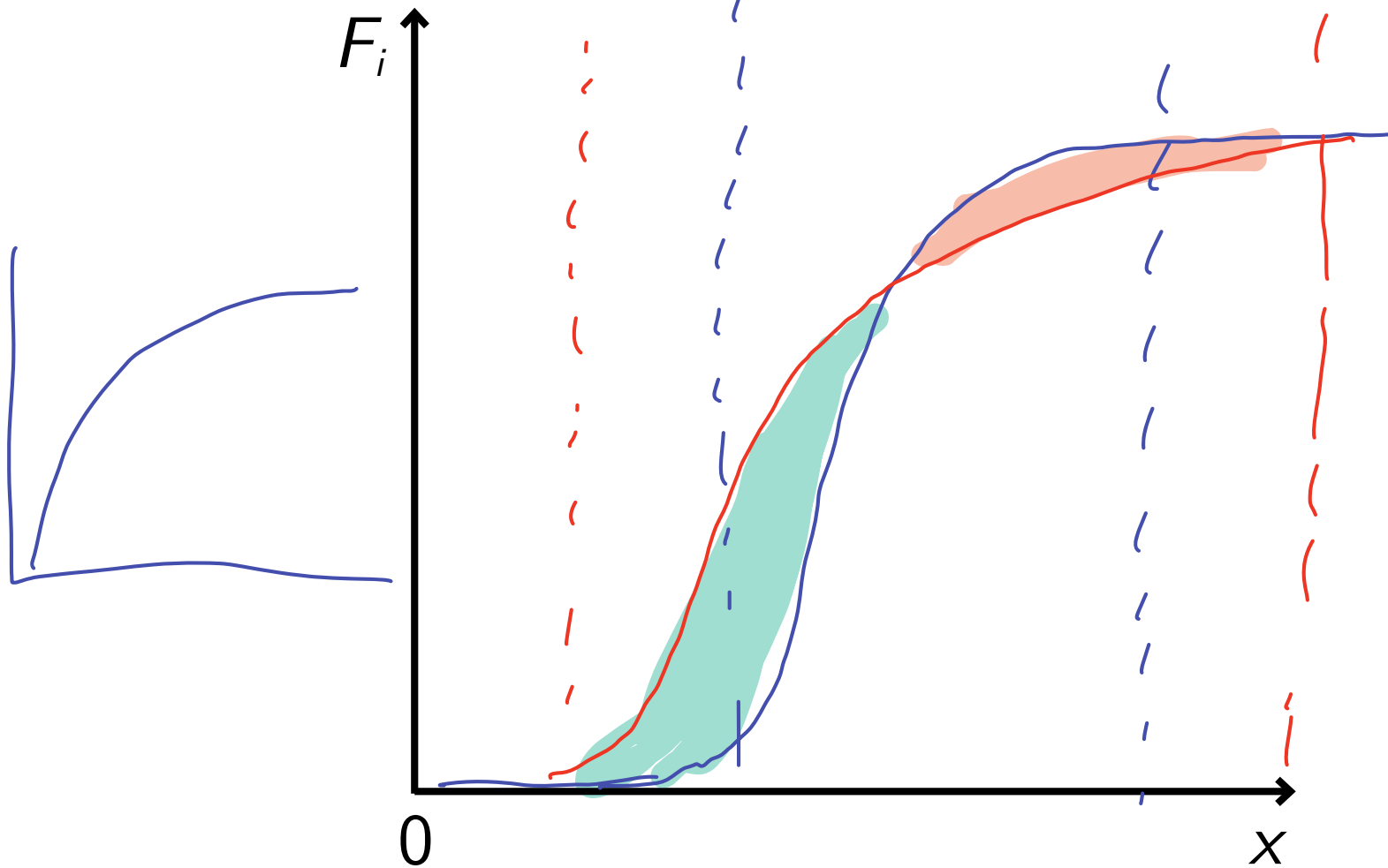
Example for FOS domination: CDFs



Example for FOS non-dominance: Densities



Example for FOS non-dominance: CDFs



3.3 Second-order stochastic dominance

Σ
∫

Definition of second-order stochastic dominance

Definition 3.2: Second-order stochastic dominance

- Let $F_A(x)$ and $F_B(x)$ be two continuously differentiable cumulative distribution functions.
- Then F_B is said to **second-order stochastically dominate** F_A iff

$$\forall x \in [a; b] : \int_a^b F_B(x) dx \leq \int_a^b F_A(x) dx$$

SOSD

and

$$\exists x \in \mathbb{R} : \int_a^b F_B(x) dx < \int_a^b F_A(x) dx.$$

FOSD

$$\forall x \in \mathbb{R} : F_B(x) \leq F_A(x)$$

Comments:

- Note that the definition of SOSD is basically the same as that for FOSD but with integrals.
- For FOSD, distribution B needs to be better (in expected value) than distribution A for any value of x .
 - The CDFs are not allowed to cross.
- For SOSD, distribution B only needs to have an expected **cumulated** advantage over distribution A for any value of x .
 - The CDFs are allowed to cross.
 - More specifically: A is allowed to be better (in expected value) for high values of x , as long as the advantage of B for lower values of x is not more than fully compensated.
 - Reason: Ask yourselves, what could be the intuition for this result?

Comments:

- Note that the definition of SOSD is basically the same as that for FOSD but with integrals.
- For FOSD, distribution B needs to be better (in expected value) than distribution A for any value of x .
 - The CDFs are not allowed to cross.
- For SOSD, distribution B only needs to have an expected **cumulated** advantage over distribution A for any value of x .
 - The CDFs are allowed to cross.
 - More specifically: A is allowed to be better (in expected value) for high values of x , as long as the advantage of B for lower values of x is not more than fully compensated.
 - Reason: Risk aversion implies that the advantage for low x is more important than the disadvantage for high x .

SOSD theorem

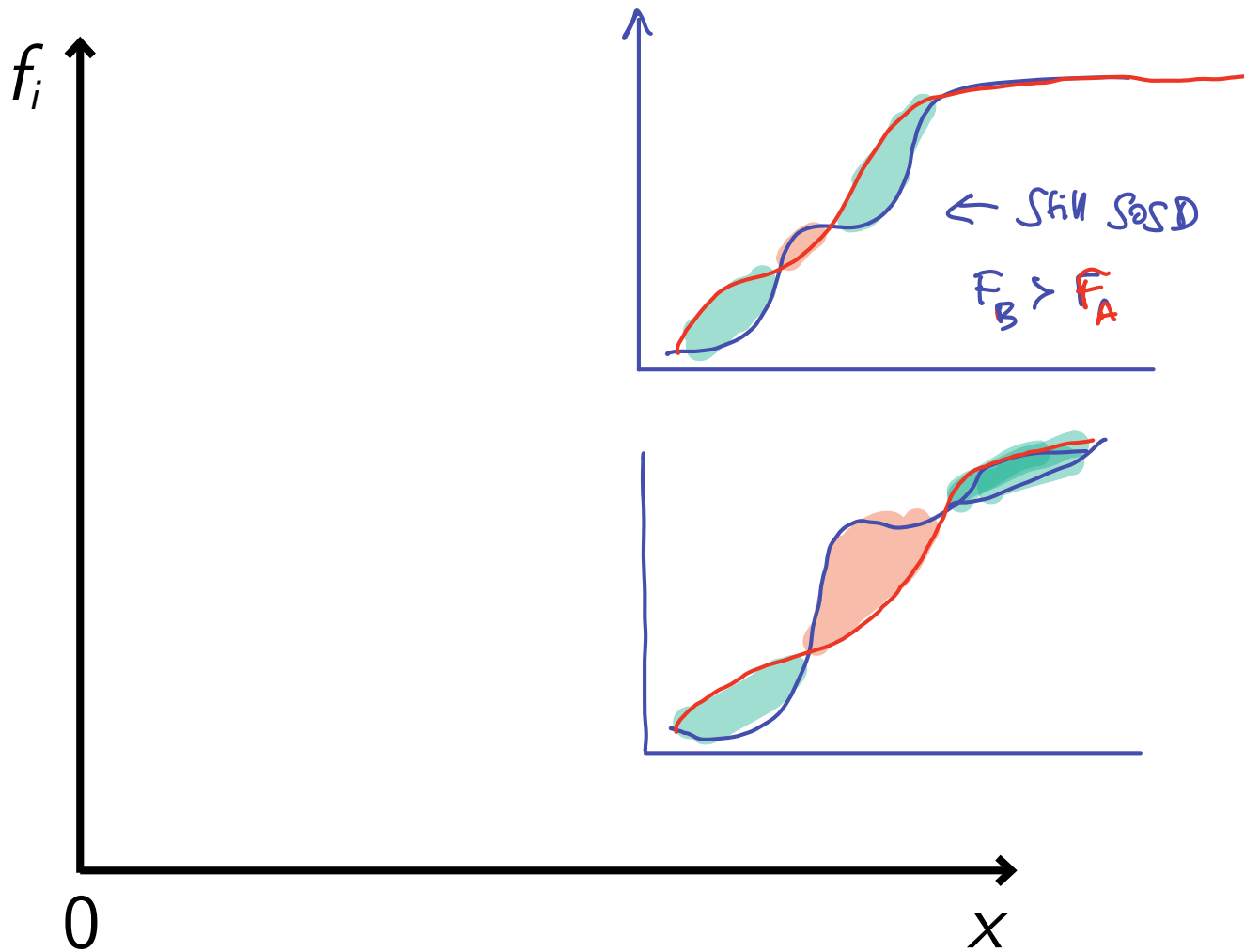
Theorem 3.2: SOSD theorem

- 1 Risk-averse** individuals with a positive marginal utility in income **prefer the second-order stochastically dominating distribution** of income.
- This implies that SOS dominated distributions do not have a **higher expected value** than SOS dominating distributions (Necessary, but not sufficient condition for SOSD).

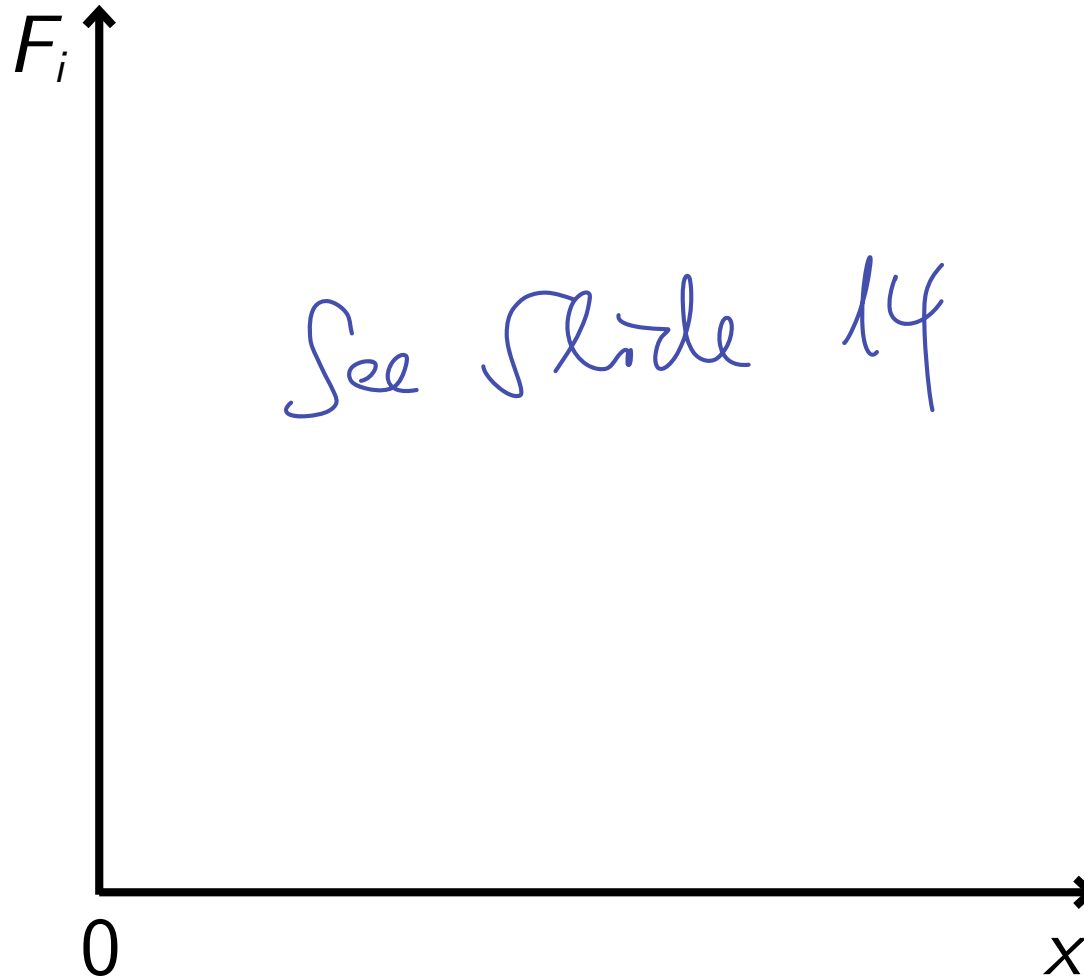
$$F_B(x) \succ^{SOSD} F_A(x) \implies E_B[x] \geq E_A[x]$$

Example for SOS domination: Densities

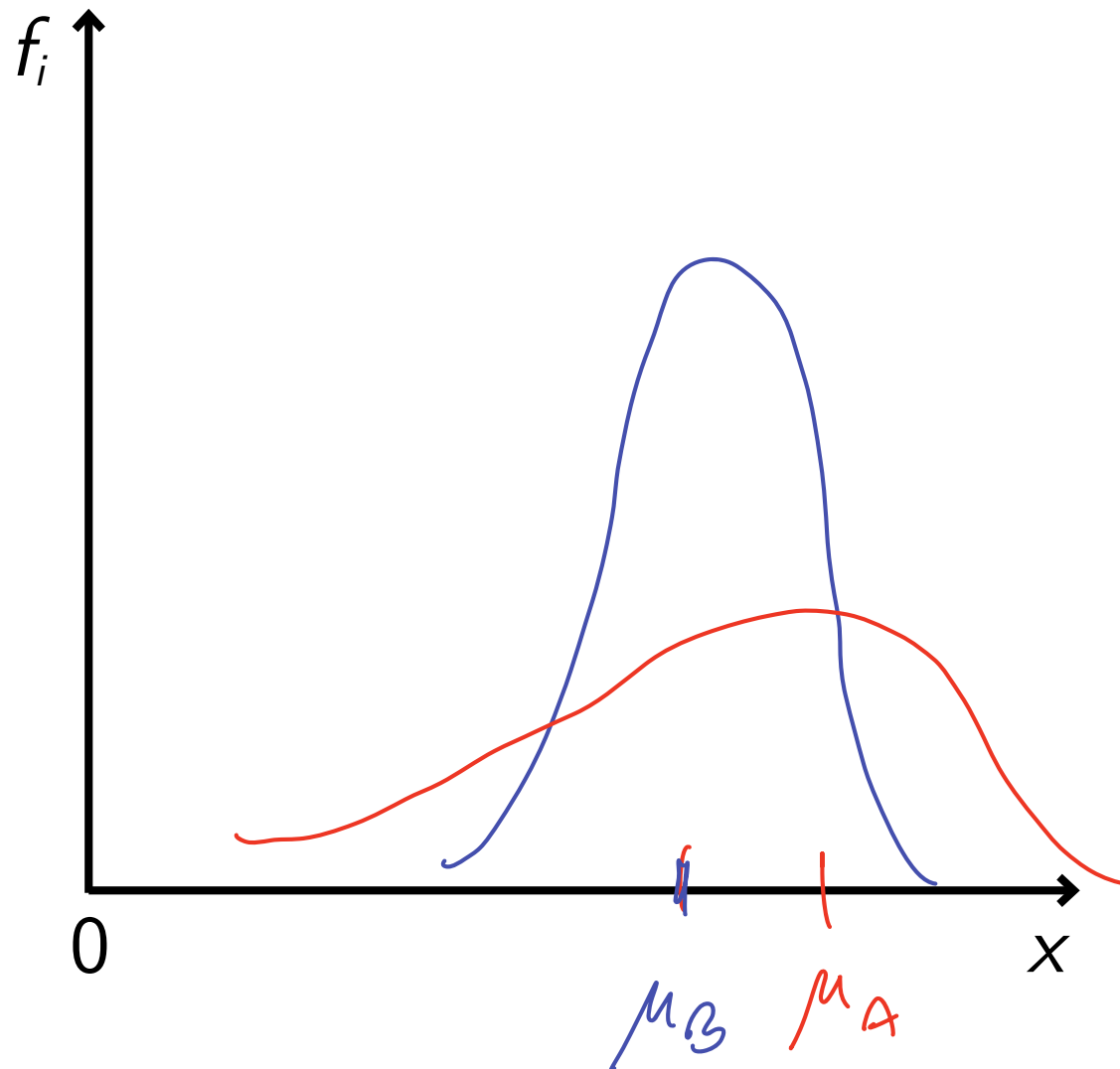
See slide 13



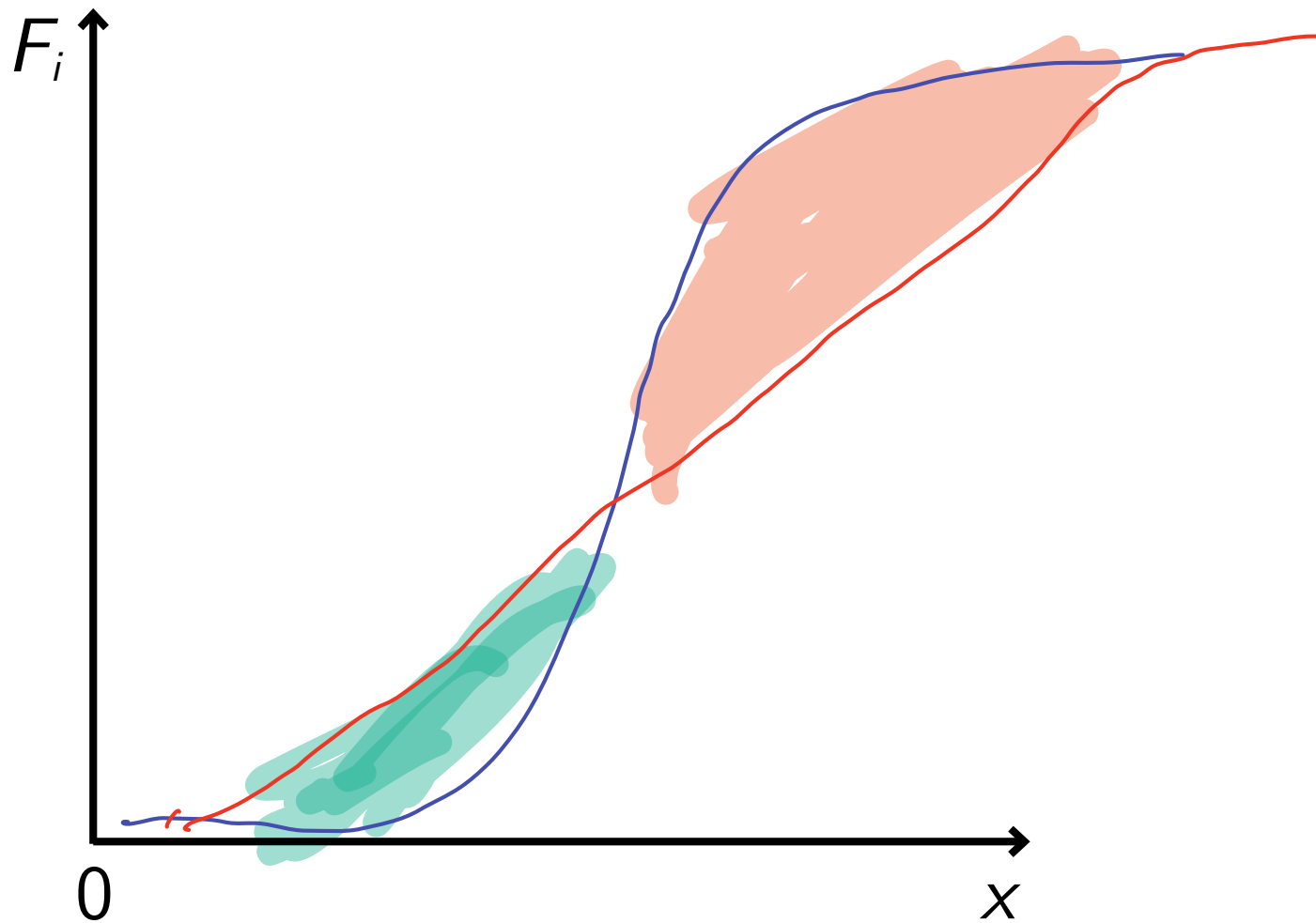
Example for SOS domination: CDFs



Example for SOS non-dominance: Densities



Example for SOS non-dominance: CDFs



Mean-preserving spread

Definition 3.3: Mean-preserving spread

$F_A(x)$ is said to be a **mean-preserving spread (MPS)** of $F_B(x)$ iff

$$F_B \succ^{SOSD} F_A$$

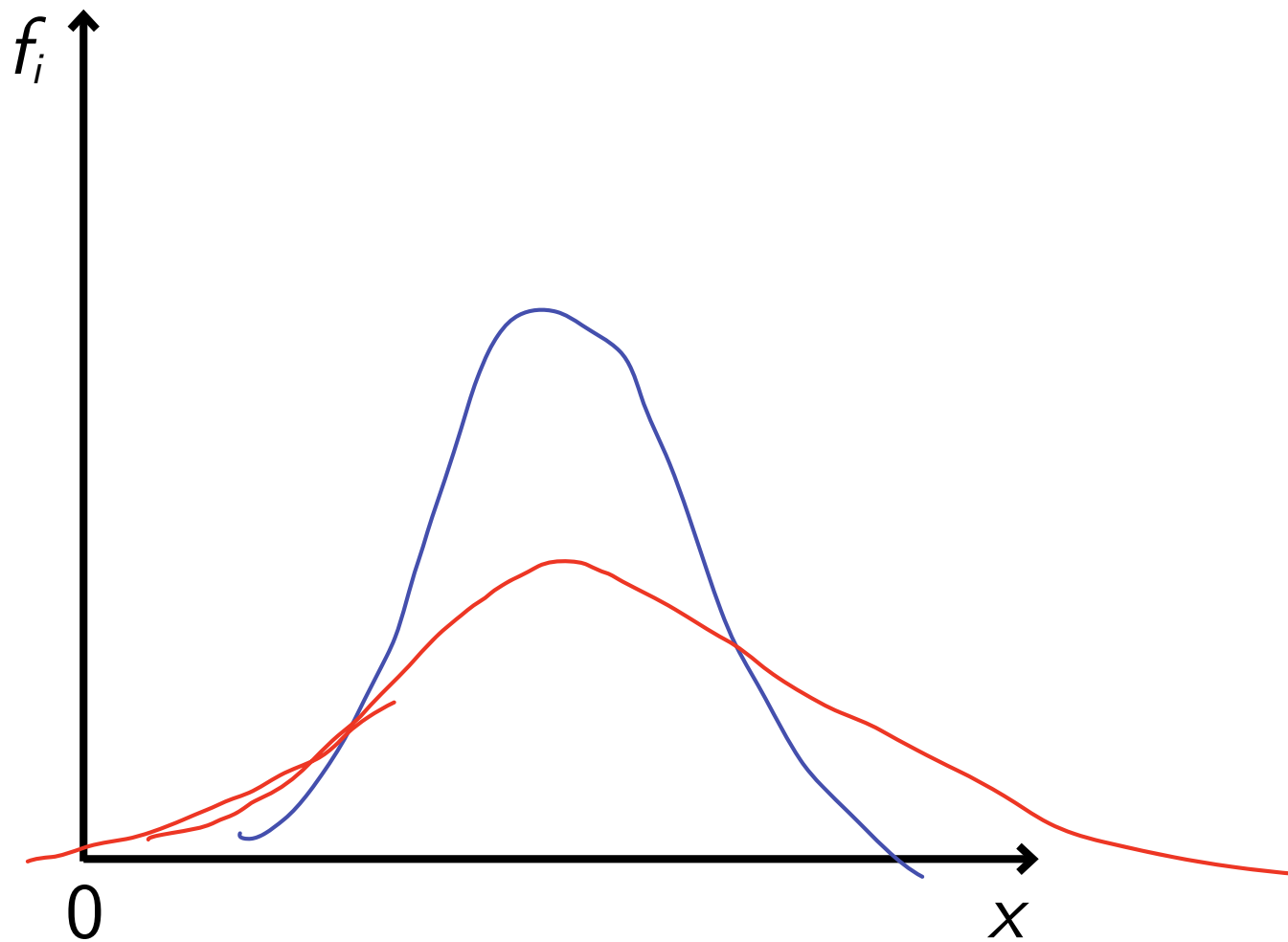
and

$$E_A(x) = E_B(x).$$

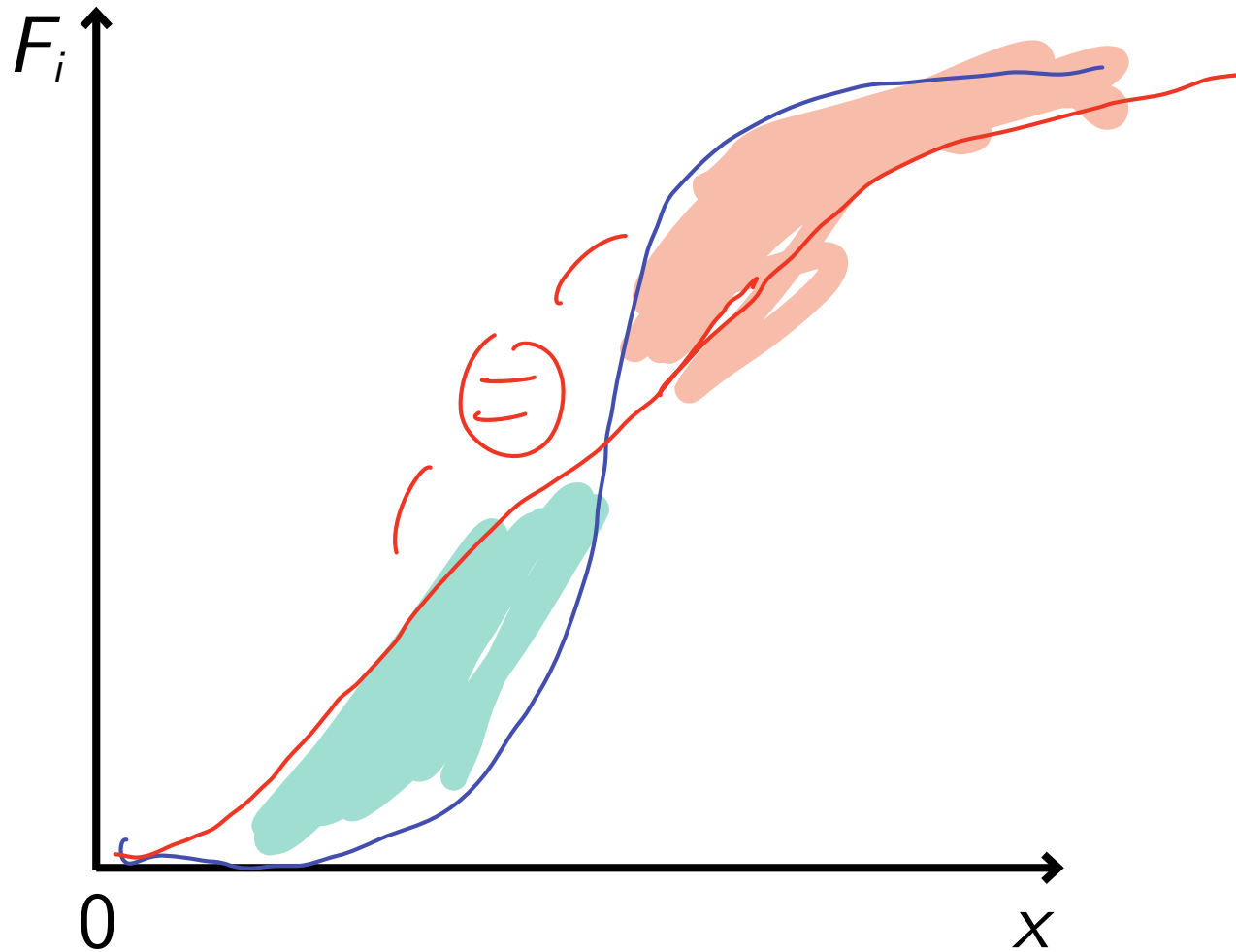
Comments:

- Note that a MPS is the border case between SOSD and non-SOSD.
- If $F_A(x)$ is a MPS of $F_B(x)$, then $F_A(x)$ has a higher variance than $F_B(x)$.

Example for MPS: Densities



Example for MPS: CDFs



The Rothschild-Stiglitz theorem (1970)

The Rothschild-Stiglitz theorem

Let there be two lotteries over $x \in [a; b]$, \mathbf{L}_A and \mathbf{L}_B , with $E_A(x) = E_B(x)$. **The following statements are equivalent:**

- 1 Any and every risk-averse agent will prefer lottery \mathbf{L}_B over \mathbf{L}_A .
- 2 $\forall x \in [a; b] : \int_a^x (F_A(u) - F_B(u)) du \geq 0$.
- 3 \mathbf{L}_A is a MPS of \mathbf{L}_B .
- 4 \mathbf{L}_A is equal to \mathbf{L}_B but for addition of white noise.

Comments:

- That $[2] \Rightarrow [1]$, we have already seen above. We will not prove $[1] \Rightarrow [2]$ here.
- $[2] \Leftrightarrow [3]$ is true by the very definition of MPS.
- $[3] \Leftrightarrow [4]$, because $[4]$ is just a different way of describing a MPS.
- Let us prove $[4] \Rightarrow [1]$.

Proof of [4] \Rightarrow [1]

- Let lottery \mathbf{L}_A be defined over $y \in [a; b]$, and lottery \mathbf{L}_B be defined over $x \in [a; b]$.
- Define white noise as ϵ : $y = x + \epsilon$, with $E[\epsilon | x] = 0$
- Show that both distributions have the same mean
 - $E_y[y] = E_{x,\epsilon}[x + \epsilon] = E_x[E_\epsilon[(x + \epsilon) | x]] = E_x[x]$
- Show that any risk-averse individual would prefer \mathbf{L}_B over \mathbf{L}_A .
 - $E_y[u(y)] = E_{x,\epsilon}[u(x + \epsilon)] \iff$
 - $E_y[u(y)] = E_x[E_\epsilon[u(x + \epsilon) | x]] \iff$
 - $E_y[u(y)] < E_x[u(x + E_\epsilon[\epsilon | x])] = E_x[u(x)] \iff$
 - $\mathbf{L}_A \prec \mathbf{L}_B$.
- QED.