

# Economic Foundations and Applications of Risk

Part B. Applications  
Chapter 7: Allocation of Risk

Till Stowasser

LMU, 2024

# Syllabus

- 7.1 Introduction
- 7.2 Efficient risk allocation
- 7.3 Arrow securities

## 7.1 Introduction

- In this chapter, we move from the individual perspective to the **social perspective on risk**.
- We will **characterize the efficient allocation of risk** between individuals if trade of state-dependent income were possible (7.2).
- Then we will introduce the concept of an **Arrow security**, which allows the aforementioned trade of risk (7.3).

## 7.2 Efficient risk allocation

Arrow / Debreu: Welfare Theorems  
↳

## Setup

- Let there be a simple exchange economy with **two individuals** (1 and 2) ...  $j = (1, 2)$
- ... and two possible **states of the world** (a and b) that realize with probabilities  $p$  and  $1 - p$ , respectively.  $i = (a, b)$
- Individual  $j$ 's **initial endowment** in state  $i$  is  $w_{0j}(i)$ , ...
- ... and her **final wealth** (i.e. after trade) in state  $i$  is  $w_{fj}(i)$ .
- Individual  $j$ 's **utility** is given by the at least twice differentiable function  $u_j$ , with  $u'_j > 0 \geq u''_j$ .

# Social planner's problem

$$\max_{w_{f1}(a), w_{f1}(b), w_{f2}(a), w_{f2}(b)} pu_1(w_{f1}(a)) + (1-p)u_1(w_{f1}(b)) \quad \text{s.t.}$$

$$pu_2(w_{f2}(a)) + (1-p)u_2(w_{f2}(b)) \geq pu_2(w_{02}(a)) + (1-p)u_2(w_{02}(b)) \equiv \bar{u}$$

$$\begin{aligned} w_{f1}(a) + w_{f2}(a) &\leq w_{01}(a) + w_{02}(a) \equiv w_0(a) \\ w_{f1}(b) + w_{f2}(b) &\leq w_0(b) \end{aligned} \quad \left. \begin{array}{l} \text{Budget} \\ \text{Constraints} \end{array} \right\}$$

↔

$$\max_{w_{f2}(a), w_{f2}(b)} pu_1(w_0(a) - w_{f2}(a)) + (1-p)u_1(w_0(b) - w_{f2}(b)) \quad \text{s.t.}$$

$$pu_2(w_{f2}(a)) + (1-p)u_2(w_{f2}(b)) \geq \bar{u}$$

## Optimal risk allocation

- Lagrangian maximization yields the following **FOC**:

$$\frac{pu'_1(a)}{(1-p)u'_1(b)} = \frac{pu'_2(a)}{(1-p)u'_2(b)}$$

- As always: The SOC holds because of risk aversion.
- This result also holds in the general case and is referred to as the **Borch condition**:

- An allocation of risk is Pareto-efficient iff, in all possible states of the world, the marginal rate of substitution of income in state  $s$  and income in state  $t$  is the same for all individuals:

$$\forall i, j, s, t : \frac{p_s u'_i(w_{fi}(s))}{p_t u'_i(w_{fi}(t))} = \frac{p_s u'_j(w_{fj}(s))}{p_t u'_j(w_{fj}(t))}$$

## Some properties of optimal risk allocation

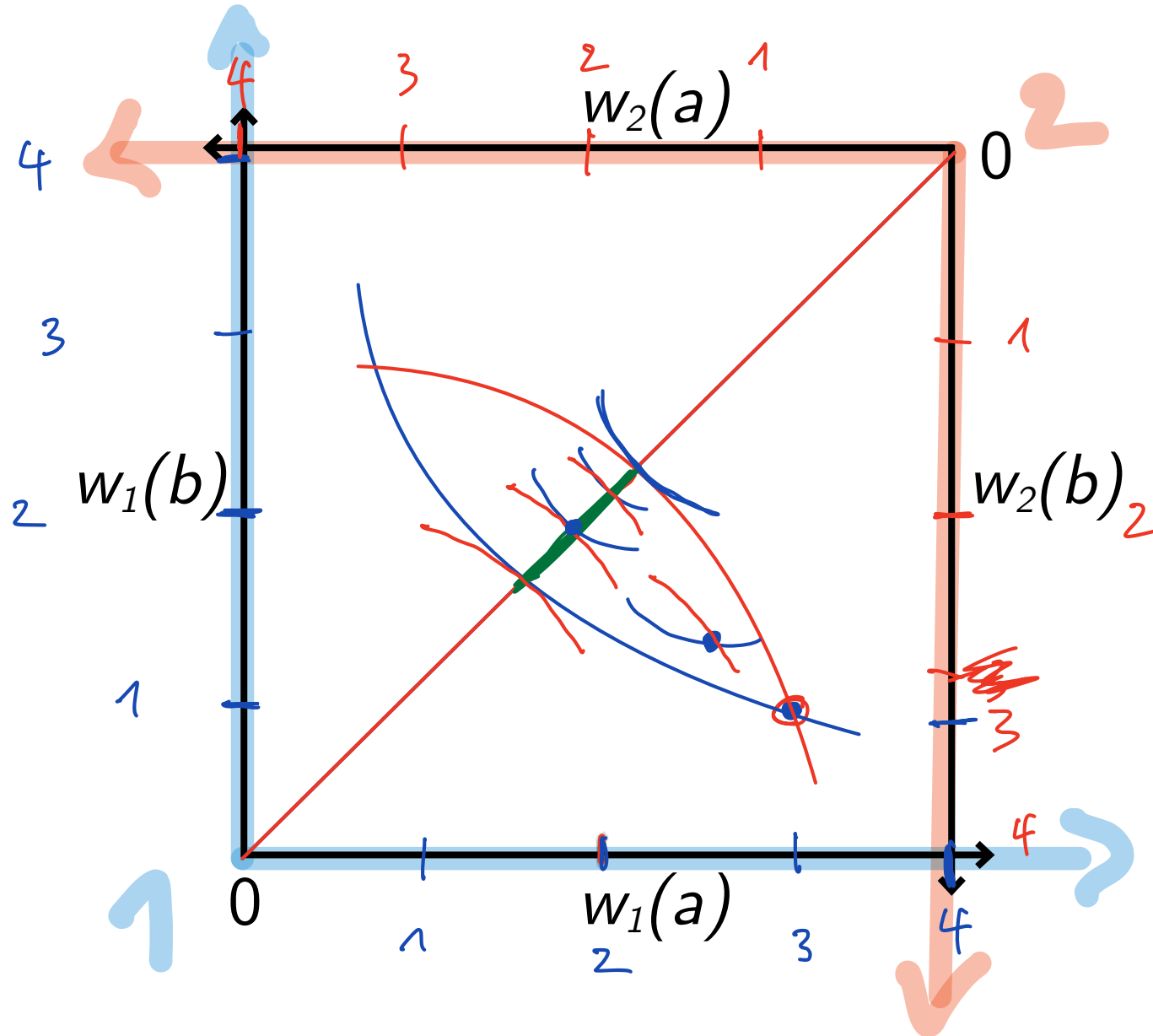
- The optimal allocation of risk between two individuals **depends** (among other things) **on** ...
- ... the **risk aversion** of individuals ...
- ... and the presence of **social risk**.
  - Social risk: For society as a whole, one state of the world is better than the other:  $w_0(a) \neq w_0(b)$  (with  $w_0(i) \equiv w_{01}(i) + w_{02}(i)$ ).
- Let us examine the following **four cases**:
  - 1 Both risk-averse & no social risk
  - 2 Both risk-averse & social risk
  - 3 One risk-neutral, one risk-averse & no social risk
  - 4 One risk-neutral, one risk-averse & social risk

## Ad 1: Both risk-averse & no social risk

- With risk being allocated optimally, **no individual will bear any risk** if ...
  - ... both individuals are risk-averse, and
  - ... there is no social risk.
- This is quite intuitive...



# Ad 1: Both risk-averse & no social risk (Graph)



	a	b
1	3	1
2	1	3
$\Sigma$	4	4

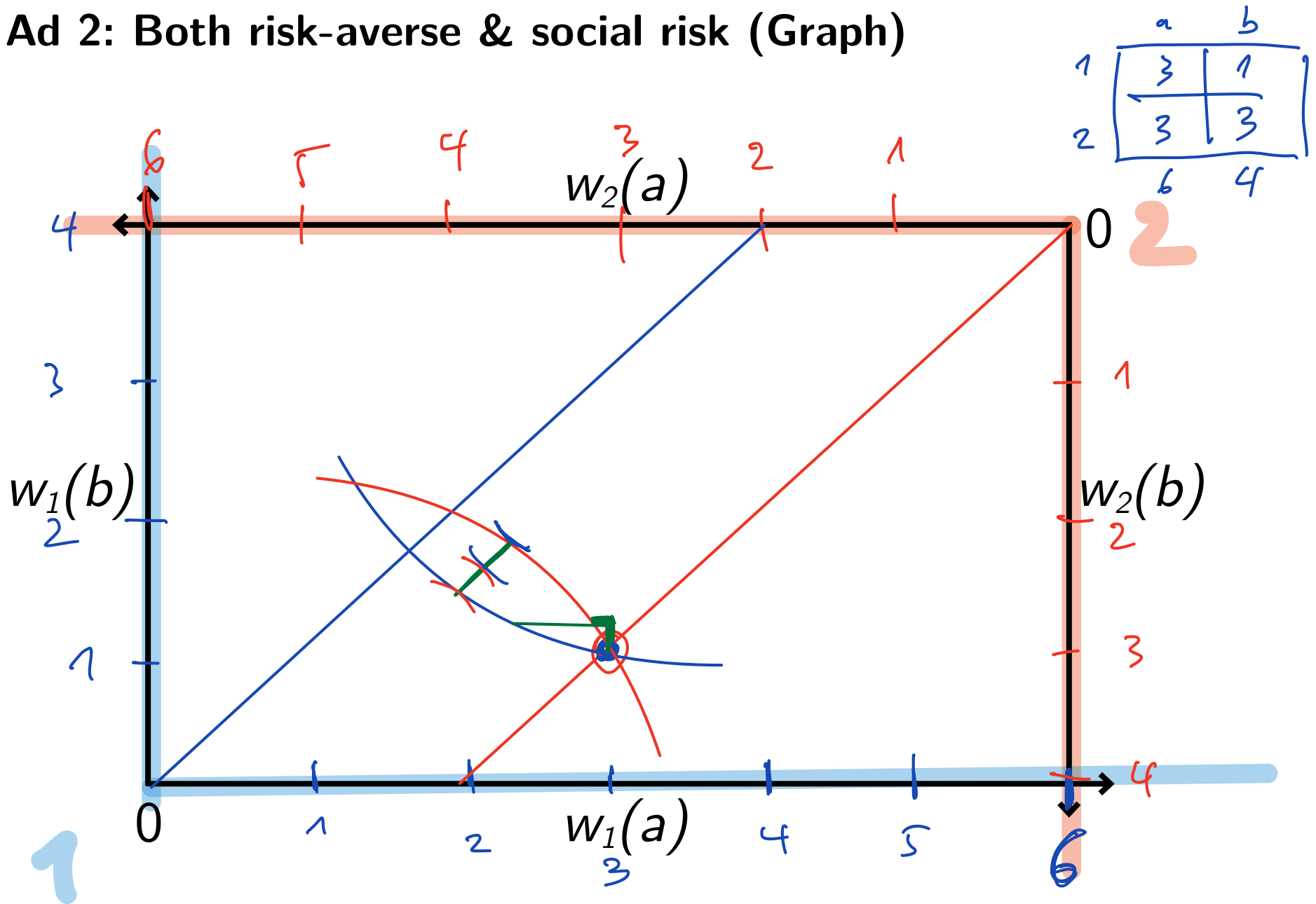
## Ad 2: Both risk-averse & social risk

- With risk being allocated optimally, **both individuals will bear some risk** if ...
  - ... both individuals are risk-averse, and
  - ... there is social risk.
- This also is quite intuitive...or is it???

	a	b
1	3	1
2	3	3
$\Sigma$	6	4

1 0.5

# Ad 2: Both risk-averse & social risk (Graph)

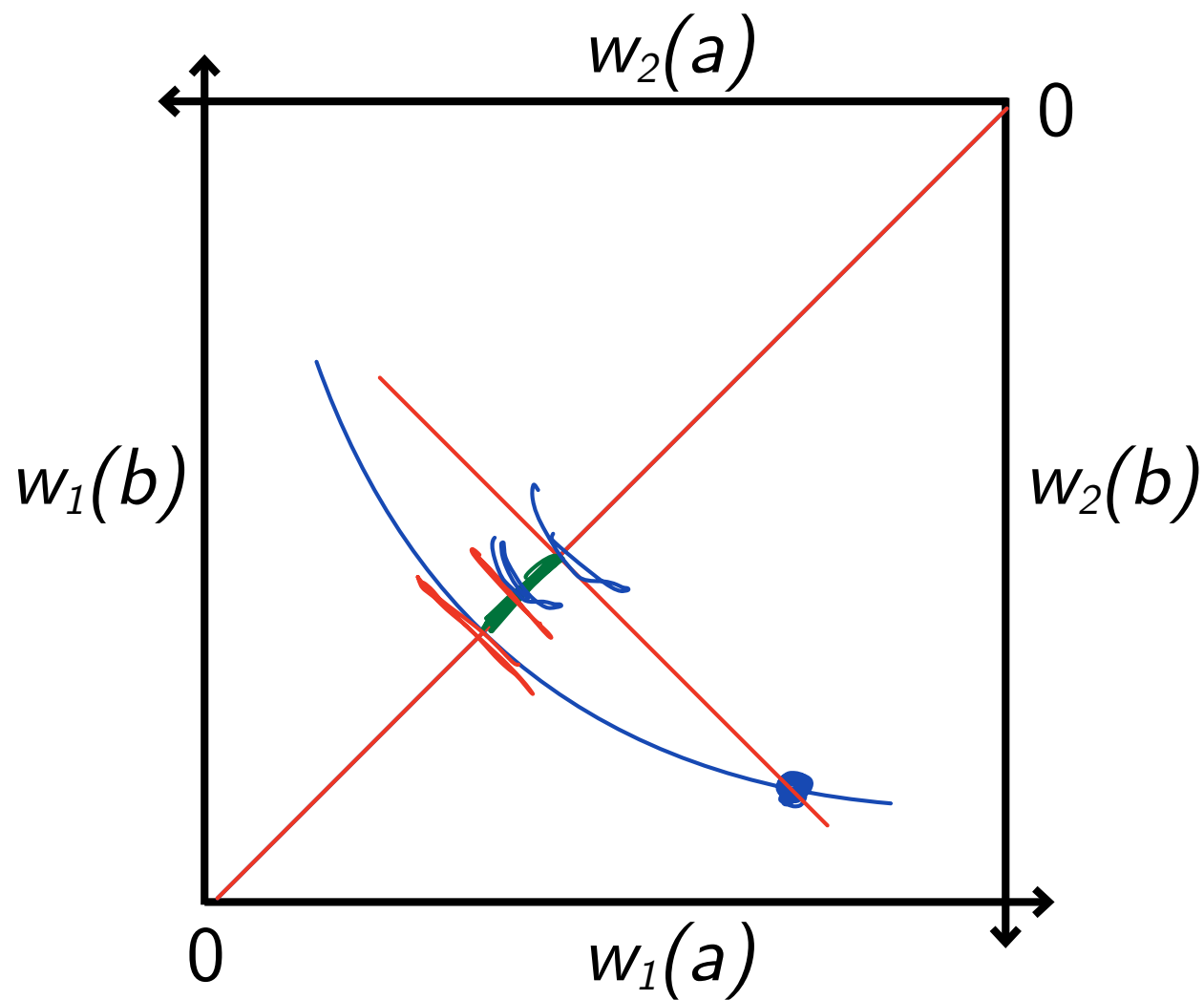


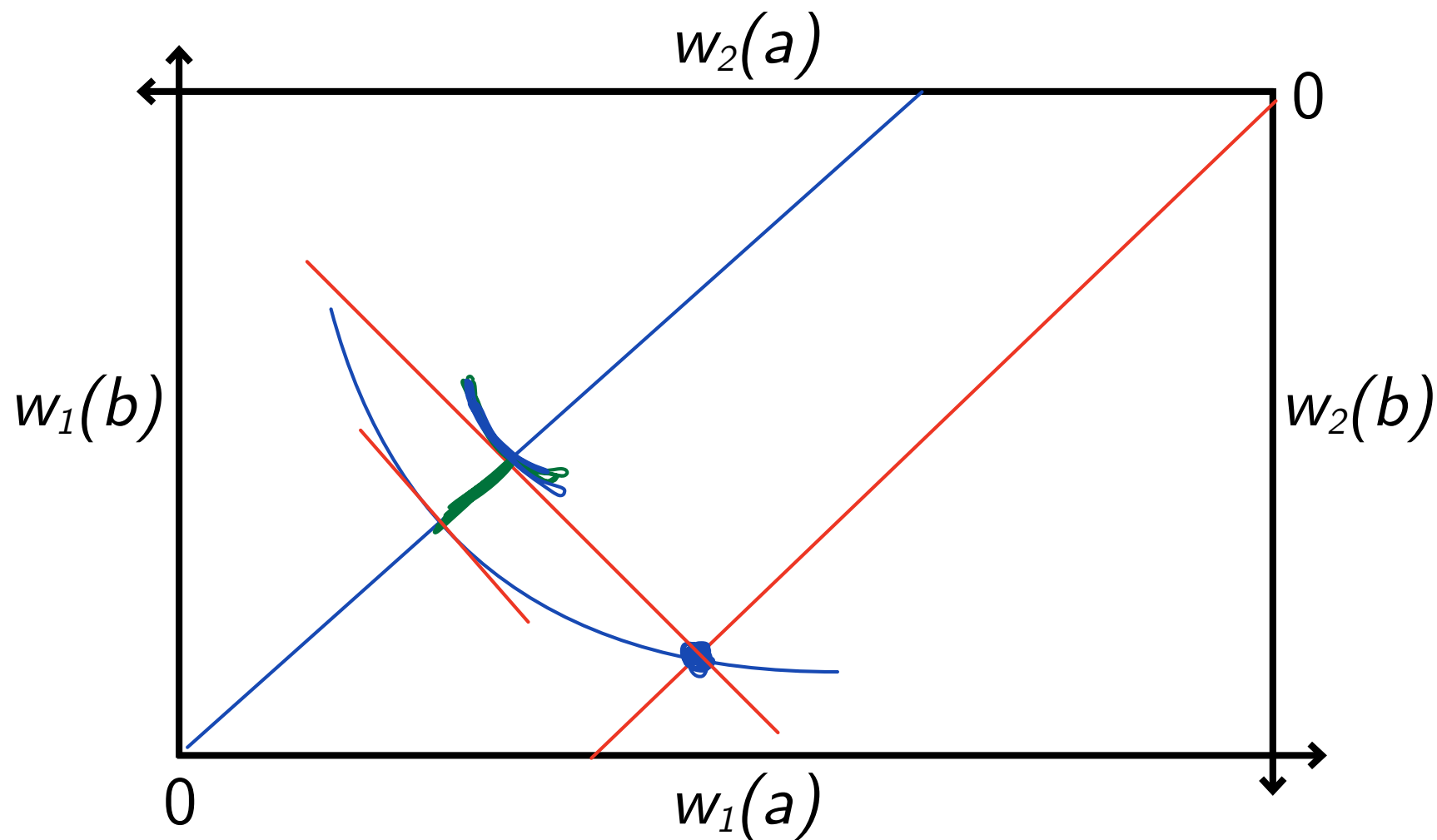
## Ad 3 and 4: One risk-neutral, one risk-averse

- With risk being allocated optimally, **the risk-neutral individual will bear all the risk** if one individual is risk-averse and the other risk-neutral.
- This also is quite intuitive...
- **Proof:**
  - Suppose individual 2 is risk-neutral (implying that  $u'_2 = \text{const}$ ).
  - Then, the Borch condition simplifies to:

$$\frac{pu'_1(a)}{(1-p)u'_1(b)} = \frac{pu'_2(a)}{(1-p)u'_2(b)} = \frac{p}{1-p}.$$

- Since individual 1 is risk-averse,  $u'_1(a) = u'_1(b) \Rightarrow w_{f1}(a) = w_{f1}(b)$ .
- Hence, the risk-neutral agent will fully insure the risk-averse agent.

**Ad 3: One risk-neutral, one risk-averse & no social risk (Graph)**

**Ad 4: One risk-neutral, one risk-averse & social risk (Graph)**

## 7.3 Arrow securities

### Motivation

- **Direct trade** of state-dependent income between individuals is **not realistic**.
- How can efficient risk-allocations be achieved?
- Arrow's idea: **Hypothetical asset** that is traded on the financial market which allows to move income along states: Arrow securities

## Setup

- An **Arrow security**  $a_s$  for state of the world  $s$  is **defined as** a security that pays out unity in state of the world  $s$  and 0 in all other states of the world.
- Let  $q_s$  denote the **market price** of one Arrow security for state  $s$ .
  - If we assume competitive markets and there is no discounting of future payments, arbitrage will lead to  $\sum_s q_s = 1$ .
- Let  $x_{fj}(s)$  denote individual  $j$ 's **final wealth** in state of the world  $s$ , which results from her initial endowment as well as from trading with Arrow securities.



## Optimization

- Individuals will maximize

$$\sum_s p_s u(x_{fj}(s)) \quad \text{s.t.}$$

$$\sum_s q_s (x_{fj}(s) - x_{0j}(s)) \leq 0$$

- First-order conditions:

$$\text{FOC for state } s: p_s u'(x_{fj}(s)) - \lambda_j q_s \stackrel{!}{=} 0$$

$$\text{FOC for state } t: p_t u'(x_{fj}(t)) - \lambda_j q_t \stackrel{!}{=} 0$$

- Dividing both FOCs yields:

$$\frac{p_s u'(x_{fj}(s))}{p_t u'(x_{fj}(t))} = \frac{q_s}{q_t}$$

- So: The market will lead to a result where  $MRS = \text{price ratio}$ .
- Hence, in equilibrium, marginal rates of substitution will be the same and the **Borch condition will hold**.
- Thus, with complete markets for Arrow securities, the **market will replicate the benevolent social planner's solution**.

## Arrow securities in the real world

- In the real world there is no market designated for Arrow securities.
- However, **many real-world markets are equivalent** to a market for Arrow securities (e.g. insurance markets, equity markets, etc.)
- Hence, regular **financial markets will lead to an efficient risk allocation if there are enough linearly independent assets.**